7. SAZONOV V.V., Periodic solutions of differential equations with a large parameter, describing the motion of generalized-conservative mechanical systems, Izv. Akad. Nauk SSSR, MTT, 3, 56-65, 1986.

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ALGEBRAIC OPERATIONS COMPATIBLE WITH THE DYNAMICS OF A NON-LINEAR DISCRETE CONTROL SYSTEM*

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A new approach is developed to the analysis and synthesis of non-linear discrete control systems

$$x[k+1] = f(k, x[k], u[k]), \quad x \in R^n, \quad u \in R^m \tag{0.1}$$

first proposed in /1/ for continuous non-linear systems. The underlying idea of the approach is to redefine the addition of state and control vectors and multiplication of vectors by scalars in such a way that the system becomes linear in the new linear space. As an application, a description is given of a class of non-linear control systems which are isomorphic to their linear approximations, and explicit formulae for this isomorphism are presented. This makes it possible to construct a control with prescribed dynamic characteristics for the linear approximation system, using the well-developed theory of the linear case; this control is then converted via the isomorphism into a control for the non-linear system, generating the required closed-loop dynamics of the system, by introducing linear feedback that compensates for the non-linearity of the open-loop system.

1. The equation for the compatibility of the addition law in $R^m \times R^n$ with the system dynamics. We seek a composition law $\oplus_x{}^k$ on the set R^n in the form of a mapping $R^{2n} \to R^n$:

$$x'' = x \oplus_x{}^k x' \stackrel{\text{def}}{=} \varphi(k, x, x') \tag{1.1}$$

Here k is a parameter and x, x' the independent variables.

Similarly, $\oplus_u{}^k$: $R^{2m} \to R^m$:

$$u'' = u \oplus_u{}^k u' \stackrel{\text{def}}{=} \psi(k, x, x', u, u') \tag{1.2}$$

Here k, x, x' are parameters and u, u' the independent variables.

Finally, $\oplus^k$: $R^{2(m+n)} \to R^{m+n}$:

$$(u, x) \oplus^k (u', x') \stackrel{\text{def}}{=} (u \oplus_u{}^k u', x \oplus_x{}^k x'), \ (u, x) \in R^{m+n} \tag{1.3}$$

Whenever there is no need to specify k we omit the superscript k of $\oplus^k$.

Let W denote the set of all pairs of functions $u(k)$, $x(k)$, $a < k < b$, satisfying (0.1). Infinite end-points $a = -\infty$, $b = \infty$ are admissible. For any $(u[k], x[k])$, $(u'[k], x'[k]) \in W$, we define

$$(u[k], x[k]) \oplus (u'[k], x'[k]) \stackrel{\text{def}}{=} (u[k] \oplus_u u'[k], \ x[k] \oplus_x x'[k]) \tag{1.4}$$

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Compatibility of $\oplus$ with the dynamics of (0.1) means that $\oplus$ does not take us out of W, i.e., $x''[k+1] = f(k, x''[k], u''[k])$. In view of (1.1), we obtain

$$\varphi(k+1, f(k, x, u), f(k, x', u')) = f(k, \varphi(k, x, x'), \psi(k, x, x', u, u')) \tag{1.5}$$

Definition 1.1. We shall say that a composition law $\oplus$ defined by (1.3) is compatible with the dynamics of system (0.1) at a time k if $\varphi(k, x, x')$, $\varphi(k+1, x, x')$, $\psi(k, x, x', u, u')$ satisfy (1.5) for all $x, x' \in R^n$, $u, u' \in R^m$.

2. The equation for the compatibility of the multiplication of control and state vectors by a scalar with the dynamics of a system.

We define $\odot_x$, $\odot_u$ as mappings $R^{n+1} \to R^n$, $R^{m+1} \to R^m$:

$$x' = \lambda \odot_x{}^k x \stackrel{\text{def}}{=} p(k, \lambda, x), \quad u' = \lambda \odot_u{}^k u \stackrel{\text{def}}{=} q(k, x, \lambda, u) \tag{2.1}$$

Here λ, x and λ, u are the independent variables in the first and second definitions, respectively. We define $\odot$ as a mapping $R^{m+n+1} \to R^{m+n}$:

$$\lambda \odot^k (u, x) \stackrel{\text{def}}{=} (\lambda \odot_u{}^k u, \lambda \odot_x{}^k x) \tag{2.2}$$

The compatibility of $\odot$ with the dynamics of system (0.1) means that if $(u[k], x[k]) \in W$, then $\lambda \odot (u[k], x[k]) \in W$, i.e., $x'[k+1] = f(k, x'[k], u'[k])$. By (2.1), this gives

$$p(k+1, \lambda, f(k, x, u)) = f(k, p(k, \lambda, x), q(k, x, \lambda, u)) \tag{2.3}$$

Definition 2.1. We shall say that the composition law $\odot$ defined by (2.2) is compatible with the dynamics of system (0.1) at a time k if $p(k, \lambda, x)$, $p(k+1, \lambda, x)$, $q(k, x, \lambda, u)$ satisfy (2.3) for all $\lambda \in R$, $x \in R^n$, $u \in R^m$.

We shall refer to (1.5) and (2.3) as the *compatibility equations*.

We now consider the relationship between a redefinition of algebraic operations compatible with the system dynamics and a linearizing change of variables. Suppose we have a linear control system

$$y[k+1] = A[k]y[k] + B[k]v[k], \quad x \in R^n, \quad v \in R^m \tag{2.4}$$

and there are given bijective mappings $x \to y$, $u \to v$ $(y = H(k, x),\ v = V(k, x, u))$ taking any pair $y[k], v[k]$ to a pair $x[k] = H^{-1}(k, y[k])$, $u[k] = V^{-1}(k, x[k], v[k])$ satisfying (0.1), where H^{-1}, V^{-1} are the inverses of H, V. Then the operations defined by the formulae

$$x'' = x \oplus_x{}^k x' = H^{-1}(k, H(k, x) + H(k, x'))$$
$$x^* = \lambda \odot_x{}^k x = H^{-1}(k, \lambda H(k, x))$$
$$u \oplus_u{}^k u' = V^{-1}(k, x'', V(k, x, u) + V(k, x', u'))$$
$$\lambda \odot_u{}^k u = V^{-1}(k, x^*, \lambda V(k, x, u))$$

are compatible with the dynamics of system (0.1). It turns out that under very general assumptions the converse is also true: given redefined operations $\oplus, \odot$, one can construct a bijective transformation H, V of system (0.1) to the form of (2.4). The proof of this statement will occupy us for the rest of this section.

Let us confine our attention to an autonomous system

$$x[k+1] = f(x[k], u[k]) \tag{2.5}$$

Let E denote the set R^n with the operations $\oplus_x$, $\odot_x$, and G the set of all pairs (x, u) i.e., the set R^{n+m}, with the operations $\oplus, \odot$. For an autonomous system (2.5) the operations $\oplus_x, \odot_x, \oplus_u, \odot_u, \oplus, \odot$ are assumed to be independent of k. In the linear spaces E, G, compatibility of the operations $\oplus, \odot$ with the dynamics of (2.5) means that the mapping $f\colon G \to E$ is linear.

Theorem 2.1. Let G, E be linear spaces, of dimensions $n+m$ and n, respectively; let $f\colon G \to E$ be a linear mapping. Then there exists a one-to-one coordinate transformation

$$y = \begin{vmatrix} \lambda_1(x) \\ \cdots \\ \lambda_n(x) \end{vmatrix}, \quad v = \begin{vmatrix} \lambda_{n+1}(x, u) \\ \cdots\cdots \\ \lambda_{n+m}(x, u) \end{vmatrix} \tag{2.6}$$

bringing system (2.5) to the linear form

$$y[k+1] = Ay[k] + Bv[k] \tag{2.7}$$

Here A and B are $n \times n$ and $n \times m$ matrices, respectively. Throughout the rest of this section we assume that the assumptions of Theorem 2.1 hold.

For the zero element $\Theta \in G$ we have

$$\begin{vmatrix} \lambda \odot_u \Theta_u \\ \lambda \odot_x \Theta_x \end{vmatrix} \equiv \lambda \odot \Theta \equiv \Theta \stackrel{\text{def}}{=} \begin{vmatrix} \Theta_u \\ \Theta_x \end{vmatrix}$$

Hence $\lambda \odot_x \Theta_x \equiv \Theta_x$, i.e., Θ_x is the zero element of E. Choose a basis $e_1, \ldots, e_n$ of E and fix u_*.

Lemma 2.1. The vectors $g_1 = (e_1, u_*), \ldots, g_n = (e_n, u_n)$ are linearly independent.

Proof. Suppose the contrary: there exist λ_i, $\Sigma \lambda_i^2 \neq 0$ $(1 \leqslant i \leqslant n)$ such that $\oplus_{i=1}^n \lambda_i \odot (e_i, u_*) = \theta$, so that

$$\begin{vmatrix} \cdots \\ \oplus_{xi=1}^n \lambda_i \odot_x e_i \end{vmatrix} = \theta$$

Therefore $\oplus_{xi=1}^n \lambda_i \odot e_i = \theta_x$, contrary to the linear independence of e_i. Here $\oplus_{i=1}^n$ denotes summation from $i = 1$ to $i = n$.

Each vector $x \in E$ is uniquely expressible as a sum $x = \oplus_{xi=1}^n \lambda_i(x) \odot_x e_i$. Define a mapping h: $E \to G$ by

$$h(x) = \oplus_{i=1}^n \lambda_i(x) \odot (e_i, u_*)$$

Lemma 2.2. h is a linear mapping.

Proof. For

$$x = \oplus_{xi=1}^n \lambda_i(x) \odot_x e_i, \quad x' = \oplus_{xi=1}^n \lambda_i(x') \odot_x e_i$$

we have

$$x \oplus_x x' = [\oplus_{xi=1}^n [\lambda_i(x) + \lambda_i(x')] \odot_x e_i$$

$$h(x \oplus_x x') = \oplus_{i=1}^n [\lambda_i(x) + \lambda_i(x')] \odot (e_i, u_*)$$

$$h(x) \oplus h(x') = [\oplus_{i=1}^n \lambda_i(x) \odot (e_i, u_*)] \oplus [\oplus_{i=1}^n \lambda_i(x') \odot (e_i, u_*)]$$

Hence $h(x \oplus_x x') = h(x) \oplus h(x')$.

Define operators $G \to G$ by the formulae

$$P_x(x, u) \stackrel{\text{def}}{=} h(x), \quad P_u(x, u) \stackrel{\text{def}}{=} (x, u) \oplus (-1) \odot P_x(x, u)$$

The following proposition is a corollary of the definition of $h(x)$ and the above lemmas (see also /2/).

Lemma 2.3. P_x and P_u are projections, and G is the direct sum of the linear spaces $P_x(G)$ and $P_u(G)$.

In $P_u(G)$ there exist m linearly independent vectors $g_{n+1}, \ldots, g_{n+m}$. The vectors g_i $(1 \leqslant i \leqslant n + m)$ form a basis of G. Any vector $(x, u) \in G$ is uniquely expressible as a sum

$$(x, u) = \oplus_{i=1}^{n+m} \lambda_i(x, u) \odot g_i \tag{2.8}$$

Lemma 2.4. $\lambda_i(x, u) = \lambda_i(x)$ for $1 \leqslant i \leqslant n$.

Proof. By Lemma 2.3,

$$P_x(x, u) = \oplus_{i=1}^n \lambda_i(x, u) \odot g_i$$

By the definitions,

$$P_x(x, u) = h(x) = \oplus_{i=1}^n \lambda_i(x) \odot g_i$$

Since the g_i are linearly independent, this completes the proof.

Proof of Theorem 2.1. Write the action of f on G in terms of matrices $A = [a_{ji}]$, $B = [b_{ji}]$:

$$f(g_i) = \oplus_{j=1}^n a_{ji} \odot_x e_j, \quad 1 \leqslant i \leqslant n$$

$$f(g_i) = \oplus_{j=1}^n b_{ji} \odot_x e_j, \quad n + 1 \leqslant i \leqslant n + m.$$

Since f is linear on G, Lemma 2.4 and (2.8) imply that system (2.5) can be rewritten as

$$\oplus_{xj=1}^n \lambda_i(x[k+1]) \odot_x e_j = x[k+1] = \oplus_{j=1}^n \Lambda_j \odot_x e_j$$

$$\Lambda_j = \sum_{i=1}^n a_{ji} \lambda_i(x[k]) + \sum_{i=n+1}^{n+m} b_{ji} \lambda_i(x[k], u[k])$$

Since e_j $(1 \leqslant j \leqslant n)$ form a basis in E, this system is equivalent to the system

$$\lambda_j(x[k+1]) = \Lambda_j, \quad 1 \leqslant j \leqslant n$$

and this, by (2.6), is equivalent to (2.7).

It should be noted that investigation of the compatibility equations is evidently more convenient than a direct search for a linearizing change of variables. For example, this approach has enabled us to develop an explicit description of a whole class of control systems admitting of linearizing transformations (see below).

3. Conditions for the solvability of the compatibility equations. We will confine our attention henceforth to the following system, which is linear in the control:

$$x[k+1] = X^\circ(k, x[k]) + Y^\circ(k, x[k])\, u[k],\ x \in R^n,\ u \in R^m \tag{3.1}$$

Fix $k \in (a, b-1)$ and let $\Gamma_x = \operatorname{Im} Y^\circ(k, x[k])$ denote the image of R^m under the linear mapping $Y^\circ(k, x[k]): R^m \to R^n$. Eq. (1.5) for (3.1) is

$$\varphi(k+1,\ X^\circ(k, x) + Y^\circ(k, x)\, u,\ X^\circ(k, x') + Y^\circ(k, x')\, u') = X^\circ(k, \varphi(k, x, x')) + Y^\circ(k, \varphi(k, x, x'))\, \psi(k, x, x', u, u') \tag{3.2}$$

Definition 3.1. We shall say that Eq. (3.2) is solvable at a time k if there exist functions $\varphi(k, x, x')$, $\varphi(k+1, x, x')$ of $x, x' \in R^n$ and a function $\psi(k, x, x', u, u')$ of x, x', u, u' satisfying Eq. (3.2) for all $x, x' \in R^n$, $u, u' \in R^m$.

Put

$$X^\circ(k, x) + \Gamma_x = \bigcup_{y \in \Gamma_x} [X^\circ(k, x) + y]$$

Lemma 3.1. Eq. (3.2) is solvable at time k if and only if there exist functions $\varphi(k, x, x')$, $\varphi(k+1, x, x')$ of $x, x' \in R^n$ such that for all $x, x' \in R^n$

$$\varphi(k+1,\ X^\circ(k, x) + \Gamma_x,\ X^\circ(k, x') + \Gamma_{x'}) - \varphi(k+1, X^\circ(k, x), X^\circ(k, x')) \in \Gamma_{\varphi(k,x,x')} \tag{3.3}$$

$$\varphi(k+1,\ X^0(k, x),\ X^\circ(k, x')) - X^\circ(k, \varphi(k, x, x')) \in \Gamma_{\varphi(k,x,x')} \tag{3.4}$$

Proof. Suppose that Eq. (3.2) is solvable. Then if $u = u' = 0$ and $u^\circ(k, x, x') = \psi(k, x, x', 0, 0) \in R^m$

$$\varphi(k+1, X^\circ(k, x), X^\circ(k, x')) = X^\circ(k, \varphi(k, x, x')) + Y^0(k, \varphi(k, x, x'))u^\circ(k, x, x') \tag{3.5}$$

This implies (3.4). Subtracting (3.5) from (3.2), we see that for $u, u' \in R^m$

$$\varphi(k+1, X^\circ(k, x) + Y^\circ(k, x)u, X^\circ(k, x') + Y^\circ(k, x')u') - \varphi(k+1), X^\circ(k, x), X^\circ(k, x')) = Y^\circ(k, \varphi(k, x, x'))[\psi(k, x, x', u, u') - u^\circ(k, x, x')] \tag{3.6}$$

This implies (3.3).

Conversely, suppose that (3.3), (3.4) hold. Then by (3.4) there exists a function $u^\circ(k, x, x')$ satisfying (3.5). Adding (3.3) and (3.5), we obtain $\varphi(k+1, X^\circ(k, x) + \Gamma_x, X^\circ(k, x') + \Gamma_{x'}) - X^\circ(k, \varphi(k, x, x')) \subset \Gamma_{\varphi(k, x, x')}$. Hence there exists a function ψ satisfying Eq. (3.2).

Henceforth we shall consider only the special case of system (3.1) in which the coefficients of the controls are constants:

$$x[k+1] = X^\circ(k, x[k]) + Y^\circ u[k], \quad 0 \leqslant m < n \tag{3.7}$$

$$Y^\circ = \begin{vmatrix} Y \\ \cdots \\ 0 \end{vmatrix}, \quad Y = \begin{vmatrix} Y_{11}, \ldots, Y_{1m} \\ \cdots\cdots \\ Y_{m1}, \ldots, Y_{mm} \end{vmatrix}, \quad \operatorname{rank} Y = m \tag{3.8}$$

If $Y = Y(k, x)$ and $\operatorname{rank} Y(k, x) = m$ then, introducing a new control $v = Y(k, x)\, u$, we reduce system (3.1) to theform of (3.7), (3.8).

Eq. (3.2) for system (3.7), (3.8) can be rewritten in the form

$$\varphi(k+1,\ X^\circ(k, x) + Y^\circ u,\ X^\circ(k, x') + Y^\circ u') = X^\circ(k,\ \varphi(k, x, x')) + Y^\circ \varphi(k, x, x', u, u') \tag{3.9}$$

Theorem 3.1. Eq. (3.9) is solvable at time k if and only if there exist functions $\varphi(k, x, x') \in R^n$, $\varphi(k+1, x, x') \in R^n$ of x, x' such that the coordinates $\varphi_i(k+1, x, x')$ with $m+1 \leqslant i \leqslant n$ are independent of the components $x_j, x_j', 1 \leqslant j \leqslant m$, $x, x' \in X^\circ(k, R^n)$, i.e.,

$$\varphi_i(k+1, x, x') = \varphi_i(k+1, x_{m+1}, \ldots, x_n, x_{m+1}', \ldots, x_n') \tag{3.10}$$

$m+1 \leqslant i \leqslant n$, $x, x' \in X^\circ(k, R^n)$, and satisfy the equations

$$\varphi_i(k+1, X^\circ(k,x), X^\circ(k,x')) = X_i^\circ(k, \varphi(k,x,x')), \; m+1 \leqslant i \leqslant n; \; x, x' \in R^n \tag{3.11}$$

Proof. By (3.8), $\Gamma = \Gamma_x$ is the linear space spanned by the first m coordinate axes. Therefore (3.3) is equivalent to the equalities

$$\varphi_i(k+1, X^\circ(k,x) + \Gamma, X^\circ(k,x') + \Gamma) - \varphi_i(k+1, X^\circ(k,x), X^\circ(k,x')) = 0, \; m+1 \leqslant i \leqslant n \tag{3.12}$$

This in turn is equivalent to (3.10). Similarly one shows that the inclusion (3.4) for system (3.7), (3.8) is equivalent to (3.11). Hence the statement of the theorem follows from Lemma 3.1.

We now consider the multiplication law $\odot$. Eq.(2.3) for system (3.1) is

$$p(k+1, \lambda, X^\circ(k,x) + Y^\circ(k,x)u) = X^\circ(k, p(k,\lambda,x)) + Y^\circ(k, p(k,\lambda,x))\, q(k,x,\lambda,u) \tag{3.13}$$

Definition 3.2. We shall say that Eq.(3.13) is solvable at time k if there exist functions $p(k,\lambda,x), p(k+1,\lambda,x), q(k,x,\lambda,u)$ of λ, x, u satisfying (3.13) for all $\lambda \in R$, $x \in R^n$, $u \in R^m$.

Arguments similar to those used above for addition yield the following propositions.

Lemma 3.2. Eq.(3.13) is solvable at a time k if and only if there exist functions $p(k, \lambda, x)$, $p(k+1,\lambda,x)$ of λ, x such that for all $\lambda \in R, x \in R^n$

$$p(k+1, \lambda \; X^\circ(k,x) + \Gamma_x) - p(k+1, \lambda, X^\circ(k,x)) \in \Gamma_{p(k,\lambda,x)}$$
$$p(k+1,\lambda, X^\circ(k,x)) - X^\circ(k, p(k,\lambda,x)) \in \Gamma_{p(k,\lambda,x)} \tag{3.14}$$

Eq.(3.13) for system (3.7), (3.8) is

$$p(k+1,\lambda, X^\circ(k,x) + Y^\circ u) = X^\circ(k, p(k,\lambda,x)) + Y^\circ q(k,x,\lambda,u) \tag{3.15}$$

Theorem 3.2. Eq.(3.15) is solvable at a time k if an only if there exist functions $p(k,\lambda,x)$, $p(k+1,\lambda,x)$ of x, λ such that the coordinates $p_i(k+1,\lambda,x)$ with $m+1 \leqslant i \leqslant n$ are independent of the components x_j with $1 \leqslant j \leqslant m$, $x \in X^\circ(k, R^n)$, i.e.,

$$p_i(k+1,\lambda,x) = p_i(k+1,\lambda,x_{m+1},\ldots,x_n), \quad m+1 \leqslant i \leqslant n, \quad x \in X^\circ(k,R^n) \tag{3.16}$$

and satisfy the equations

$$p_i(k+1,\lambda,X^\circ(k,x)) = X_i^\circ(k, p(k,\lambda,x)), \quad m+1 \leqslant i \leqslant n, \quad x \in R^n \tag{3.17}$$

4. Solution of the compatibility equations.

The following proposition is obvious.

Theorem 4.1. Let $H(x)$: $R^n \to R^n$ be a bijective mapping of R^n onto R^n. Then the operations of addition $\oplus_x$ and multiplication by scalars $\odot_x$ defined by the formulae

$$x \oplus_x x' \overset{\text{def}}{=} H^{-1}(H(x) + H(x')), \quad \lambda \odot_x x \overset{\text{def}}{=} H^{-1}(\lambda H(x)) \tag{4.1}$$

determine a new linear space structure on R^n, with zero element $\Theta = H^{-1}(0)$, and the mapping $x \to H(x)$ is an isomorphism of the new linear space R^n relative to $\oplus_x$, $\odot_x$ and the original Euclidean space R^n relative to ordinary vector addition $x + x'$ and multiplication by scalars λx.

We propose the following problem: define a bijective mapping $x \to H(k,x)$ in such a way that the functions defined according to (4.1)

$$\varphi(k,x,x') \overset{\text{def}}{=} H^{-1}(k, H(k,x) + H(k,x'))$$
$$p(k,\lambda,x) \overset{\text{def}}{=} H^{-1}(k, \lambda H(k,x)) \tag{4.2}$$

will satisfy (3.10), (3.11), (3.16), (3.17). Here k is a parameter and $H^{-1}(k,\cdot)$ is the inverse of the mapping $x \to H(k,x)$, i.e., $H^{-1}(k, H(k,x)) \equiv x$.

Put

$$Z(k,x) = \begin{vmatrix} Z_1(k,x) \\ \cdots\cdots \\ Z_{n-m}(k,x) \end{vmatrix} = \begin{vmatrix} X^\circ_{m+1}(k,x) \\ \cdots\cdots \\ X_n^\circ(k,x) \end{vmatrix}, \quad F(k,x) = \begin{vmatrix} X_1^\circ(k,x) \\ \cdots \\ X_m^\circ(k,x) \end{vmatrix} \tag{4.3}$$

We will confine our discussion to the simplest case, when the compatibility equations are solvable by explicit formulae for φ, p, ψ, q.

Define a mapping $H(k, x)$ by

$$H_i(k, x) \overset{\text{def}}{=} Z_i(k, x) = X_{m+i}^{\circ}(k, x), \quad 1 \leqslant i \leqslant n - m \qquad (4.4)$$
$$H_i(k, x) \overset{\text{def}}{=} x_i, \quad n - m + 1 \leqslant i \leqslant n$$

We need an additional simplifying assumption: the functions defined by (4.2) must satisfy the condition

$$\varphi_i(k + 1, x, x') = x_i + x_i', \quad p'(k + 1, \lambda, x) = \lambda x_i, \quad m + 1 \leqslant i \leqslant n \qquad (4.5)$$

In particular, condition (4.4) will hold if $n - m \leqslant m$, i.e., $n \leqslant 2m$, since in that case (4.5) follows from the last formula of (4.5). Condition (4.5) will also hold if the components $H_i(k, x)$, $m + 1 \leqslant i \leqslant n$, can be written as

$$H^{(z)}(k, x) = H_0^{(z)}(k)\, x^{(z)}, \quad H^{(z)} = \begin{vmatrix} H_{m+1} \\ \cdots \\ H_n \end{vmatrix}, \quad x^{(z)} = \begin{vmatrix} x_{m+1} \\ \cdots \\ x_n \end{vmatrix}$$

where $H_0^{(z)}(k)$ is some non-singular $(n - m) \times (n - m)$ matrix.

When condition (4.5) is satisfied, so are conditions (3.10) and (3.16), and Eqs. (3.11) and (3.17) become

$$X_i^{\circ}(k, x) + X_i^{\circ}(k, x') = X_i^{\circ}(k, \varphi(k, x, x')) \qquad (4.6)$$
$$\lambda X_i^{\circ}(k, x) = X_i^{\circ}(k, p(k, \lambda, x)), \quad m + 1 \leqslant i \leqslant n$$

which, in terms of the notation (4.3), may be rewritten as

$$Z(k, x) + Z(k, x') = Z(k, \varphi(k, x, x')), \quad \lambda Z(k, x) = Z(k, p(k, \lambda, x)) \qquad (4.7)$$

By (4.4), we see that φ and p as defined by (4.2) satisfy (4.7). Hence, in view of Theorems 3.1, 3.2 and 4.1, we obtain the following

Theorem 4.2. Let condition (3.8) hold, let $x \to H(k, x)$, $x \to H(k + 1, x)$ be bijections of R^n onto R^n and φ, p functions defined by (4.2) and satisfying condition (4.5). Then $\varphi(k, x, x')$, $p(k, \lambda, x)$ define a new linear space structure in R^n relative to the operations $\oplus_x$, $\odot_x$ defined by (1.1), (2.1), with zero element $\Theta(k) = H^{-1}(k, 0)$; the mapping $x \to H(k, x)$ is an isomorphism of the space relative to $\oplus_x, \odot_x$ and the Euclidean space R^n relative to ordinary addition $x + x'$ and multiplication by scalars λx; the compatibility Eqs. (3.9) and (3.15) are solvable at a time k, and the functions φ, p defined by (4.2) are their solutions for certain functions $\psi(k, x, x', u, u')$, $q(k, \lambda, x, u)$.

We must now determine the functions ψ and q defining a composition law for the controls in accordance with (1.2), (2.1). It is obvious that the functions φ, p defined by (4.2) satisfy Eqs. (4.7), which are just the last $n - m$ equations of systems (3.9) and (3.15). The functions ψ and q, in their turn, are found from the first m equations of systems (3.9) and (3.15), which can be expressed in the following form. Denote

$$\varphi^{(m)}(k, x, x') \overset{\text{def}}{=} \begin{vmatrix} \varphi_1(k, x, x') \\ \cdots\cdots \\ \varphi_m(k, x, x') \end{vmatrix}, \quad p^{(m)}(k, \lambda, x) \overset{\text{def}}{=} \begin{vmatrix} p_1(k, \lambda, x) \\ \cdots\cdots \\ p_m(k, \lambda, x) \end{vmatrix} \qquad (4.8)$$

Then, in view of (3.8) and (4.3), we obtain from (3.9)

$$\varphi^{(m)}(k + 1, X^{\circ}(k, x) + Y^{\circ}u, X^{\circ}(k, x') + Y^{\circ}u') = F(k, \varphi(k, x, x')) + Y\psi(k, x, x', u, u')$$

Hence, by (3.8), we can write

$$\psi(k, x, x', u, u') = Y^{-1}[\varphi^{(m)}(k + 1, X^{\circ}(k, x) + Y^{\circ}u, X^{\circ}(k, x') + Y^{\circ}u') - F(k, \varphi(k, x, x'))] \qquad (4.9)$$

Similarly, we derive from (3.15)

$$q(k, x, \lambda, u) = Y^{-1}[p^{(m)}(k + 1, \lambda, X^{\circ}(k, x) + Y^{\circ}u) - F(k, p(k, \lambda, x))] \qquad (4.10)$$

The remainder of this section is taken up with the proof that definitions (1.3) and (2.2) define a linear space structure on the set $R^m \times R^n$ relative to operations $\oplus$ and $\odot$, which is compatible with the dynamics of system (3.7), (3.8). Consider the set $\Lambda = R^{2n}$, denoting its elements by $\begin{vmatrix} x_* \\ x \end{vmatrix}$, $x_* \in R^n$, $x \in R^n$. Define laws $\oplus$ and $\odot$ on Λ by

$$\begin{vmatrix} x_* \\ x \end{vmatrix} \oplus \begin{vmatrix} x_*' \\ x' \end{vmatrix} \overset{\text{def}}{=} \begin{vmatrix} \varphi(k+1, x_*, x_*') \\ \varphi(k, x, x') \end{vmatrix} \qquad (4.11)$$

$$\lambda \odot \begin{vmatrix} x_* \\ x \end{vmatrix} \overset{\text{def}}{=} \begin{vmatrix} p(k+1, \lambda, x_*) \\ p(k, \lambda, x) \end{vmatrix}$$

Rewrite these relations in the equivalent form

$$\begin{vmatrix} x_* \\ x \end{vmatrix} \oplus \begin{vmatrix} x_*' \\ x' \end{vmatrix} \overset{\text{def}}{=} \begin{vmatrix} x_* \oplus_x^{k+1} x_*' \\ x \oplus_x{}^k x' \end{vmatrix} \qquad (4.12)$$

$$\lambda \odot \begin{vmatrix} x_* \\ x \end{vmatrix} \overset{\text{def}}{=} \begin{vmatrix} \lambda \odot_x^{k+1} x_* \\ \lambda \odot_x{}^k x \end{vmatrix}$$

The set Λ with operations (4.12) will be denoted by $\Lambda(k)$. Theorem 4.1 implies the following

Theorem 4.3. Let (3.8) hold, let $x \to H(k, x)$, $x \to H(k+1, x)$ be bijections of R^n onto R^n and φ, p functions defined by (4.2) and satisfying condition (4.5). Then the operators (4.2) give Λ the structure of a linear space $\Lambda(k)$ with zero $\begin{vmatrix} \Theta(k+1) \\ \Theta(k) \end{vmatrix}$.

Put

$$\Lambda_0(k) = \left\{ \begin{vmatrix} X^\circ(k, x) + Y^\circ u \\ x \end{vmatrix} : x \in R^n,\ u \in R^m \right\}$$

Theorem (4.2), and Eqs.(3.9) and (3.15) imply the following

Lemma 4.1. Under the assumptions of Theorem 4.3, $\Lambda_0(k)$ is a linear subspace of $\Lambda(k)$.

Let $\Omega(k)$ denote the set $R^m \times R^n$ of all pairs (u, x), $u \in R^m$, $x \in R^n$, equipped with operations (1.3) and (2.2) with functions ψ, q defined by (4.9), (4.10). Define a mapping $h: \Omega(k) \to \Lambda_0(k)$ by

$$(u, x) \to h(u, x) \overset{\text{def}}{=} \begin{vmatrix} X^\circ(k, x) + Y^\circ u \\ x \end{vmatrix} \qquad (4.13)$$

Lemma 4.2. Under the assumptions of Theorem 4.3, $\Omega(k)$ is a linear space and h is an isomorphism of $\Lambda_0(k)$ and $\Omega(k)$.

Proof. By (3.8), h is a bijection. It will therefore suffice to prove that h commutes with the operations $\oplus$ and $\odot$:

$$h((u, x) \oplus (u', x')) = h(u, x) \oplus h(u', x') \qquad (4.14)$$

$$h(\lambda \odot, u(x)) = \lambda \odot h(u, x) \qquad (4.15)$$

By (1.3), (4.13) and the fact that $\oplus_x = \oplus_x{}^k$, we obtain

$$h((u, x) \oplus (u', x')) = h((u \oplus_u u'), (x \oplus_x x')) = \begin{vmatrix} X^c(k, x \oplus_x x') + Y^c(u \oplus_u u') \\ x \oplus_x x' \end{vmatrix} \qquad (4.16)$$

On the other hand, by (4.13), (4.11),

$$h(u, x) \oplus h(u', x') = \begin{vmatrix} X^c(k, x) + Y^c u \\ x \end{vmatrix} \oplus \begin{vmatrix} X^c(k, x') + Y^c u' \\ x' \end{vmatrix} = \begin{vmatrix} \varphi(k+1, X^\circ(k, x) + Y^\circ u, X^\circ(k, x') + Y^c u') \\ x \oplus_x x' \end{vmatrix} \qquad (4.17)$$

By (4.9) and Theorem 4.2, Eq.(3.9) is satisfied. It follows from (1.2) and (3.9) that (4.16) and (4.17) are identical, implying the truth of (4.14). The proof of (4.15) is similar.

Theorem 4.4. Let condition (3.8) hold, let $x \to H(k, x)$, $x \to H(k+1, x)$ be bijections of R^n onto R^n and φ, p functions defined by (4.2) and satisfying condition (4.5). Then the operations (1.3), (2.2) with functions ψ, q defined by (4.9), (4.10) make the set of all pairs $(u, x) \in R^m \times R^n$ a linear space $\Omega(k)$, and moreover the composition laws $\oplus, \odot$ defined by (1.4), (2.3) are compatible with the dynamics of system (3.7) at a time k.

Proof. The laws $\oplus$ and $\odot$ were defined in such a way as to satisfy Eqs.(3.9) and (3.15). They are therefore compatible with the dynamics of system (3.7). That $\Omega(k)$ is a linear space follows from Lemma 4.2.

5. Linearizing isomorphism. We will now consider an isomorphism that will be used in Sect.6 to construct an isomorphism of the linear space W of the initial non-linear system onto the linear space W° of the linearized system. We define this isomorphism as the mapping $x \to y = \varkappa(k, x) \in R^n$ obtained by solving the equation

$$H_x(k, \Theta(k))\, y = H(k, x) \quad \text{when } \Theta(k) = H^{-1}(k, 0) \tag{5.1}$$

Theorem 5.1. Let the mapping $x \to H(k, x)$ be a bijection of R^n onto R^n and let us assume that the matrix of partial derivatives $H_x(k, x)$ is continuous in $x \in R^n$ and

$$\operatorname{rank} H_x(k, x) = n \quad \text{for} \quad x \in R^n \tag{5.2}$$

Then the mapping $x \to y = \varkappa(k, x)$ defined by (5.1) is a bijection of R^n onto R^n; it is uniquely defined together with its inverse

$$y \to x = s(k, y) \tag{5.3}$$

by the equations

$$H_x(k, \Theta(k))\, \varkappa(k, x) = H(k, x) \tag{5.4}$$

$$H_x(k, \Theta(k))\, y = H(k, s(k, y)) \tag{5.5}$$

and the two mappings define an isomorphism of the linear space $x \in R^n$ with zero element $\Theta(k)$, relative to the operations $\oplus_x, \odot_x$ defined by (1.1), (2.1), (4.2), and the linear space $y \in R^n$ relative to the natural operations $+, \cdot$ and zero element $y = 0$. Moreover, if $p(k, \lambda, x)$ is the function defined by (4.2), then the limits in the following formulae exist and satisfy the relations

$$\varkappa(k, x) = \lim_{\lambda \to 0} \lambda^{-1}[p(k, \lambda, x) - \Theta(k)] = \left.\frac{\partial p(k, \lambda, x)}{\partial \lambda}\right|_{\lambda = 0} \tag{5.6}$$

$$s(k, y) = \lim_{\lambda \to 0} p(k, \lambda^{-1}, \lambda y + \Theta(k)) \tag{5.7}$$

It it is assumed that condition (4.5) also hold, then

$$\Theta_i(k) = 0, \quad m + 1 \leqslant i \leqslant n \tag{5.8}$$

$$s_i(k, y) = y_i, \quad \varkappa_i(k, x) = x_i, \quad m + 1 \leqslant i \leqslant n \tag{5.9}$$

Proof. By (5.2) and Theorem 4.1, $\varkappa(k, x)$ is a composition of isomorphisms $x \to H(k, x)$, $x' \to H_x^{-1}(k, \Theta(k))\, x'$. Therefore $\varkappa(k, x)$ is also an isomorphism. Eqs.(5.4) and (5.5) follow from (5.1). By (4.2)

$$H(k, p(k, \lambda, x)) \equiv \lambda H(k, x) \tag{5.10}$$

By the Implicit Function Theorem, $p(k, \lambda, x)$ is continuously differentiable. By (5.10), $p(k, 0, x) = \Theta(k)$. Differentiate (5.10) with respect to λ at $\lambda = 0$: $H_x(k, \Theta(k))\, \partial p(k, 0, x)/\partial\lambda = H(k, x)$. In addition,

$$p(k, \lambda, x) = \Theta(k) + \lambda \frac{\partial p(k, 0, x)}{\partial \lambda} + o(\lambda), \quad \lim_{\lambda \to 0} \lambda^{-1} o(\lambda) = 0$$

The last two expression, together with (5.4), imply (5.6). Since $H(k, \Theta(k)) = 0$, it follows that $H(k, \lambda y + \Theta(k)) = H_x(k, \Theta(k))\, \lambda y + o(\lambda)$. By (4.2),

$$\lim_{\lambda \to 0} p(k, \lambda^{-1}, \lambda y + \Theta(k)) = \lim_{\lambda \to 0} H^{-1}(k, \lambda^{-1} H(k, \lambda y + \Theta(k))) = H^{-1}(k, H_x(k, \Theta(k))\, y)$$

Hence, by (5.5), we obtain (5.7). Since $\Theta(k) = p(k, 0, x)$, (4.5) implies (5.8). Then (5.9) follows from (5.6) and (5.7).

6. Synthesis of control systems using the isomorphism. Consider the linear approximation system for (3.7):

$$y[k+1] = X_x^\circ(k, \Theta(k))\, y[k] + Y^\circ v[k], \quad v \in R^m, \quad y \in R^n \tag{6.1}$$

where X_x° is the matrix of partial derivatives. Suppose that conditions of Theorem 5.1 are satisfied. We have $\varkappa(k+1, x[k+1]) = y[k+1]$. By (3.7), (6.1), this implies the fundamental equation for conversion of controls:

$$\varkappa(k+1,\ X^\circ(k,x)+Y^\circ u)=X_x^\circ(k,\Theta(k))\,\varkappa(k,x)+Y^\circ v \tag{6.2}$$

Applying the transformation $y\to s(k+1,y)$, we derive the following equivalent version of (6.2):

$$X^\circ(k,s(k,y))+Y^\circ u=s(k+1,X_x^\circ(k,\Theta(k))\,y+Y^\circ v) \tag{6.3}$$

We assert that (6.2) and (6.3) are identities in the components with indices $m+1\leqslant i\leqslant n$. By (3.8) and (5.9), we have

$$\varkappa_i(k+1,\ X^\circ(k,x)+Y^\circ u)=X_i^\circ(k,x),\quad m+1\leqslant i\leqslant n \tag{6.4}$$

From (5.4) and (4.4), we obtain

$$(X_x^\circ(k,\Theta(k))\,\varkappa(k,x))_i=X_i^\circ(k,x),\quad m+1\leqslant i\leqslant n \tag{6.5}$$

where $(\cdot)_i$ denotes the i-th coordinate. It is clear from (6.4), (6.5) and (3.8) that the vector components indexed $m+1\leqslant i\leqslant n$ on the right and left of (6.2) are identical. By (5.9) this implies the same for (6.3). Therefore, by (3.8), Eq.(6.2) is uniquely solvable in v and Eq.(6.3) is uniquely solvable in u.

Let $v(k,x,u)$, $u(k,y,v)$ denote the solutions of Eqs.(6.2) and (6.3), respectively. Consider the mappings

$$(x,u)\to(y=\varkappa(k,x),\ v=v(k,x,u)) \tag{6.6}$$

$$(y,v)\to(x=s(k,y),\ u=u(k,y,v)) \tag{6.7}$$

Since $\varkappa$ and s are inverses of one another and (6.3) is equivalent to (6.2), the mappings (6.6) and (6.7) are also inverses of one another.

Let W° denote the set of pairs of functions $(y[k],\ v[k])$, defined by (6.1) for $a<k<b$.

Theorem 6.1. Assume that (3.8) holds and that the following conditions are satisfied for $a<k<b$: 1) $X(k,x)$ is continuously differentiable with respect to $x\in R^n$; 2) the mappings $x\to H(k,x)$ are bijections of R^n onto R^n and $\operatorname{rank} H_x(k,x)=n$ for $x\in R^n$; 3) the functions φ, p defined by (4.2) satisfy condition (4.5). Then the mappings (6.6), (6.7) define mutually inverse mappings

$$W\to W^\circ\colon (x[k],u[k])\to(y[k]=\varkappa(k,x[k]),\ v[k]=v(k,x[k],u[k])) \tag{6.8}$$

$$W^\circ\to W\colon (y[k],v[k])\to(x[k]=s(k,y[k]),\ u[k]=u(k,y[k],v[k])) \tag{6.9}$$

which are isomorphisms of the linear spaces W for system (3.7) and W° for (6.1) relative to the operations (1.4), (2.3) and $+,\cdot$, respectively.

Proof. Let $x[k],u[k]$ satisfy (3.7) for $a<k<b$. We must prove that the pair $y[k]$, $v[k]$, defined by (6.8) satisfies (6.1). By (6.8), (3.7) and (6.2),

$$y[k+1]=\varkappa(k+1,x[k+1])=\varkappa(k+1,X^\circ(k,x[k])+Y^\circ u[k])=X_x^\circ(k,\Theta(k))\varkappa(k,x[k])+Y^\circ v[k]=X_x^\circ(k,\Theta(k))y[k]+Y^\circ v[k]$$

i.e., (6.8) is indeed a mapping $W\to W^\circ$. It similarly follows from (6.3) that (6.9) is a mapping $W^\circ\to W$.

Since (6.8) and (6.9) are mutually inverse mappings, it will suffice to prove that (6.8) is a linear space homomorphism $W\to W^\circ$. By Theorem 5.1, $\varkappa$ is an isomorphism; hence we have the following lemma.

Lemma 6.1. Let

$$\varkappa(k+1,x_*)=y_*,\quad \varkappa(k+1,x_x')=y_*'$$

Then

$$\varkappa(k+1,x_*\oplus_x^{k+1}x_*')=y_*+y_*',\quad \varkappa(k+1,\lambda\odot_x^{k+1}x_*)=\lambda y_*$$

Now let

$$y=\varkappa(k,x),\ v=v(k,x,u),\ y'=\varkappa(k,x'),\ v'=v(k,x',u') \tag{6.10}$$

To prove that (6.8) is a homomorphism, it will suffice to show that

$$y+y'=\varkappa(k,x\oplus_x{}^kx'),\quad v+v'=v(k,x\oplus_x{}^kx',u\oplus_u{}^ku') \tag{6.11}$$

$$\lambda y=\varkappa(k,\lambda\odot_x{}^kx),\quad \lambda v=v(k,\lambda\odot_x{}^kx,\lambda\odot_u{}^ku) \tag{6.12}$$

The first part of (6.11) follows from Theorem 5.1; so we need only prove the second. Conditions (6.10) imply that (6.2) holds and $\varkappa(k+1,\ \ X^\circ(k,x')+Y^\circ u')=X_x^\circ(k,\Theta(k))\varkappa(k,x')+Y^\circ v'$. Put $x_*=X^\circ(k,x)+Y^\circ u,\ x_*'=X^\circ(k,x')+Y^\circ u'$. Then by (6.2), putting $y_*=\varkappa(k+1,x_*), y_*'=\varkappa(k+1,x_*')$, we obtain

$$y_*=X_x^\circ(k,\Theta(k))\varkappa(k,x)+Y^\circ v,\ y_*'=X_x^\circ(k,\Theta(k))\varkappa(k,x')+Y^\circ v'.$$

By (3.9), we have $x_*\oplus_x^{k+1}x_x'=X^\circ(k,x\oplus_x^k x')+Y^\circ(u\oplus_u^k u')$. Hence, by Lemma 6.1,

$$\begin{aligned}&\varkappa(k+1,X^\circ(k,x\oplus_x^k x')+Y^\circ(u\oplus_u^k u'))=\\&\quad X_x^\circ(k,\Theta(k))[\varkappa(k,x)+\varkappa(k,x')]+Y^\circ(v+v')=\\&\quad X_x^\circ(k,\Theta(k))\varkappa(k,x\oplus_x^k x')+Y^\circ(v+v')\end{aligned}\tag{6.13}$$

At the same time, by the definition (6.2), it follows that for $v(k,x\oplus_x^k x',\ \ u\oplus_u^k u')$

$$\begin{aligned}&\varkappa(k+1,X^\circ(k,x\oplus_x^k x')+Y^\circ(u\oplus_u^k u'))=\\&\quad X_x^\circ(k,\Theta(k))\varkappa(k,x\oplus_x^k x')+Y^\circ v(k,x\oplus_x^k x',u\oplus_u^k u')\end{aligned}\tag{6.14}$$

Subtracting (6.14) from (6.13) and using (3.8), we obtain (6.11). The proof of (6.12) is analogous. Thus the mappings (6.6) and (6.7) are indeed mutually inverse isomorphisms of W and W°.

Now let

$$X^\circ(k,0)\equiv 0,\quad a<k<b\tag{6.15}$$

By (4.4), it follows that

$$\Theta[k]=0,\quad a<k<b\tag{6.16}$$

Let $k,y\to v(k,y)=D(k,y)\in R^m$ be some mapping defined for $k\geqslant k_0,\ y\in R^n$. By (6.6), (6.7), it defines a mapping $k,x\to u(k,x)=u(k,\varkappa(k,x),D(k,\varkappa(k,x)))$. As a result, we can consider systems (3.7) and (6.1) with feedbacks $u(k,y),\ v(k,y)$:

$$x[k+1]=X^\circ(k,x[k])+Y^\circ u(k,\varkappa(k,x[k]),D(k,\varkappa(k,x[k])))\tag{6.17}$$

$$y[k+1]=X_x^\circ(k,0)y[k]+Y^\circ D(k,y[k])\tag{6.18}$$

By Theorem 6.1 and the definitions we have

Corollary 6.1. Under the assumptions of Theorem 6.1, assume moreover that condition (6.15) holds, $(y[k],v[k])$ and $(x[k],u[k])$ $(a<k_0\leqslant k\leqslant k_f<b)$ are elements of W° and W paired by the isomorphisms (6.8) and (6.9). Then $y[k_f]=0$ if and only if $x[k_f]=0$. If in addition $b=\infty$, $D(k,0)\equiv 0$ for $k\geqslant k_0$ and the mappings $\varkappa(k,x)$, $s(k,y)$ are continuous uniformly in $k\geqslant k_0$ at the points $x=0$ and $y=0$, then the trivial solution of system (6.17) is stable or asymptotically stable in the large (in Lyapunov's sense) if and only if the trivial solution of system (6.18) has the corresponding property.

Remark 6.1. In /3/ we generalized the results of /1/ to non-autonomous systems, and in /4/ we considered questions of the synthesis of discrete systems.

7. Example.

Consider the non-linear control system

$$\begin{aligned}x_1[k+1]&=x_1[k]x_2[k]+u[k]\\x_2[k+1]&=(x_2^2[k]+1)x_1[k]\end{aligned}\tag{7.1}$$

This is an instance of system (3.7) with $m=1,\ n=2$. By (4.3) and (4.4),

$$H_1(x)=(x_2^2[k]+1)x_1[k],\quad H_2(x)=x_2\tag{7.2}$$

By (4.2), $H(\varphi)=H(x)+H(x')$, whence, by (7.2), we obtain

$$(\varphi_2^2+1)\varphi_1=(x_2^2+1)x_1+((x_2')^2+1)x_1',\quad \varphi_2=x_2+x_2'$$

Hence, noting (1.1), we express the law $\oplus_x$ in terms of the coordinates by

$$\begin{aligned}&(x\oplus_x x')_1\equiv\varphi_1(k,x,x')=[(x_2+x_2')^2+1]^{-1}[(x_2^2+1)x_1+((x_2')^2+\\&\quad 1)x_1']\\&(x\oplus_x x')_2=x_2+x_2'\end{aligned}$$

Similarly, by (4.2), $H(p)=\lambda H(x)$, whence it follows, by (2.1), (7.2), that

$$\begin{aligned}(\lambda\odot_x x)_1&\equiv p_1=(\lambda^2x_2^2+1)^{-1}\lambda(x_2^2+1)x_1\\(\lambda\odot_x x)_2&\equiv p_2=\lambda x_2\end{aligned}$$

The isomorphism $\varkappa$ is defined by (5.4):

$$H_x(0)\ \varkappa(x)=H(x),\quad H_x(0)=\begin{vmatrix}1&0\\0&1\end{vmatrix}$$

$$y_1=\varkappa_1(x)=(x_2^2+1)x_1,\quad y_2=\varkappa_2(x)=x_2$$

The linear approximation equations (6.1) are in this case

$$y_1[k+1] = v[k], \quad y_2[k+1] = y_1[k] \tag{7.3}$$

The first equation of system (6.2) may be written

$$\beta(x)(x_1x_2 + u) = v, \quad \beta(x) = (x_2^2+1)^2x_1^2 + 1 \tag{7.4}$$

and the second becomes an identity. Under the control $v = D(y) = ay_1 + by_2$, $a = 1$, $b = -0.25$ system (7.3) is asymptotically stable in the large. By (7.4), the isomorphism yields a control

$$u = (ay_1 + by_2)\beta^{-1}(x) - x_1x_2 = [a(x_2^2+1)x_1 + bx_2]\beta^{-1}(x) - x_1x_2$$

of system (7.1), under which, by Corollary 6.1, the trivial solution is asymptotically stable in the large. The same control is also obtainable using the following formula form /4/:

$$u = \lim_{\lambda\to 0} \lambda^{-1} \odot_u D(\lambda \odot_x x)$$

REFERENCES

1. PANASYUK A.I., The introduction of algebraic operations on the set of trajectories of a non-linear control system. Prikl. Mat. Mekh., 42, 4, 1984.
2. KOSTRIKIN A.I. and MANIN YU.L., Linear Algebra and Geometry, Nauka, Moscow, 1986.
3. PANASYUK A.I., Algebraic operations compatible with the dynamics of continuous non-linear control systems. Differents. Uravn., 24, 3, 1988.
4. PANASYUK A.I., A method of synthesizing discrete control systems by introducing algebraic operations on the set of trajectories. Izv. Vuz. Priborostroyenie, 9, 1986.

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DYNAMIC MODELLING OF UNKNOWN PERTURBATIONS IN PARABOLIC VARIATIONAL INEQUALITIES*

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The problems involved in the dynamic determination of the load acting on a membrane rigidly fixed on a horizontal frame are investigated, and the thermal flux in a thermostat is determined. These problems are treated as special cases of a more general problem: dynamic modelling of unknown characteristics in parabolic variational inequalities. The problem is solved by constructing an algorithm, stable to information noise and computing errors, based on methods of positional control theory /1, 2/. This algorithm may be regarded as a modification of an algorithm proposed in /3/ for control systems described by ordinary differential equations. A model problem is solved. The research reported, here relies on /3, 4/ and is a sequel to /5/.

1. In /6/ (Vol.1, p.198) a numerical method of determining the deflection $y(x, t)$ of a membrane rigidly fixed on a horizontal frame with constant tension F, subject to a given load $g(x, t)$ is proposed.

We shall consider the inverse problem: to determine the load $g(x, t)$ given the deflection $y(x, t)$. Let Ω be the plane region bounded by the frame. Put

$$u(x, t) = g(x, t)/F, \quad K = \{v(\cdot) \mid v(x) \in H_0^1, \quad v(x) \leqslant 0 \text{ in } \Omega\}$$
$$K_* = \{v(\cdot) \mid v(\cdot) \in L_2([t_0, \vartheta]; H_0^1), \ \partial v/\partial t \in L_2([t_0, \vartheta]; H^{-1}),$$
$$v(t_0) = y_0\}$$

$H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are Sobolev spaces. The deflection process for a membrane subject to a

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